CALCULUS 1- FUNCTIONS

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INTRODUCTION

What is Calculus?

- Calculus is the study of how things change. That is, a framework for modeling systems wherein change is present , and a way to deduce or predict that change.
- The fundamental concept of Calculus is “instantaneous change”, which means change as it occurs in over miniscule intervals of time.
- Calculus is built on the ideas of Archimedes, Fermat, Newton, Leibniz, Cauchy, and other great thinkers.

- Calculus was invented by Newton, who discovered that acceleration could be modeled by relatively simple laws of motion. That is, calculus examines the interrelations between motion, acceleration, and position.
- Calculus has two parts: **differential** and **integral** calculus.

How Calculus Evolved?

- The word “calculus” comes from the word “rock”, which means a stone. People in ancient times did math with piles of stones, a particular method of computation in math.
- Arithmetic and geometry which originated in ancient times co-existed for over seven hundred years as separate branches of math.
- Rene Descartes, a French philosopher and mathematician was able to unite the analytical power of algebra with the descriptive power of geometry- which is analytic geometry.

- In about 1670, Calculus was discovered by Sir Isaac Newton of England. However, the symbols of Gottfried Wilhelm Leibniz of Germany were used in Calculus.
- Calculus was developed out of a need to understand continuously changing quantities.

Why Study Calculus?

- As a math/science major, you are required to take it.
- It is considered to be one of the most important branches of math. That is, it is different from arithmetic, algebra, trigonometry and geometry because of the concept “limit” of a function.

- It is a powerful tool for solving a wide variety of mathematical problems in the fields such as physical, biological and social sciences.
- It is an indispensable tools in the fields of economics, oceanography, meteorology, medicine, space science, psychology, physics, chemistry, hydraulic/electrical engineering, sports equipment manufacturing, stock market analysis and physiology.

- Calculus is the mathematics of self-actualization. That is, understanding who you are, understanding yourself in relation to yourself and to others(functions), understanding the difference between your actual limits and the limits imposed upon you by others(limits), understanding the changes that you have experienced and have to undergo(derivatives), assimilating the changes within you and re-connecting them with who you are(integration)

- Calculus is a journey to find yourself. Just as Calculus is the bridge between algebra, geometry , and the higher math. Calculus is also the bridge between you and the real life.

Chapter 1: FUNCTIONS

1.1 Definition of a Function
1.2 Properties of Graphs of Functions
1.3 Types of Functions & Operations on Functions
1.4 Functions as Mathematical Models

1.1 DEFINITION OF A FUNCTION

Cartesian Product

- DEFINITION 1. Let A and B be sets. The set
- $A \times B = \{(a, b): a \epsilon A \ and \ b \epsilon B\}$is the Cartesian Product (cross product) of A and B
- EXAMPLE 1. If A={1,2,3} and B= {a,b}, find A x B.
- EXAMPLE 2. If A = {3,4} and B is the closed interval [2,3].
- (a) model this in a Cartesian plane,

  (b) find Ax B.

Relation

- DEFINITION 2. A relation $\Re$ between sets A and B is a subset of A x B.
- EXAMPLE 3. Determine relations using Ex. 1
- EXAMPLE 4. Find relations using Ex. 2.

Functions

- We shall now consider graphs, ordered pairs, equations and diagrams in defining functions.

Rule Form of the Definition of a Function

- DEFINITION 3.

A **function** is a rule that produces a correspondence between two sets of elements such that to each element in the first set there corresponds **one and only one** element in the second set. The set of all first element s is called the **domain**, and the set of all corresponding elements in the second set is **range**.

EXAMPLE 5. Which of the following tables specify a function/s?

1.

Domain (number)	Range (square)

2.

Domain (number)	Range (absolute value)

3.

Domain (number)	Range ($\pm$ *square roots*)

Set Form of the Definition of a Function

- DEFINITION 4.

A **function** is a set of ordered pairs with the property that no two ordered pairs have the same first component and different second components.

The set of all first component is called the **domain**, and the set of all second components is called the **range**.

EXAMPLE 6. Which set defines a function?

1. $S=\{(1,4), (2,3), (3,2), (4,3), (5,4)\}$
2. $T= \{(1,4), (2,3), (3,2), (2,4), (1,5)\}$
3. $I = \{(1,1), (2,2), (3,3), (4,4)\}$

Function Defined by Equation

- DEFINITION 5.

In an equation in two variables, if each value of the independent variable there corresponds exactly one value of the dependent variable, then the equation defines a **function**.

If there is any value of the independent variable to which there corresponds more than one value of the dependent variable, then the equation does **not define a function**.

EXAMPLE 7. Determine which equations define a function?

Note: x is the independent variable, and domain is the set of real numbers.

1. $y^3 - x = 1$
2. $y^2 - x^2 = 9$

Graph of a Function

- DEFINITION 6. The graph of a function is the graph of all ordered pairs that belong to the function.
- EXAMPLE 8. Graph each function. State the domain and the range of each function.

$$1. f(x) = |x - 2|$$

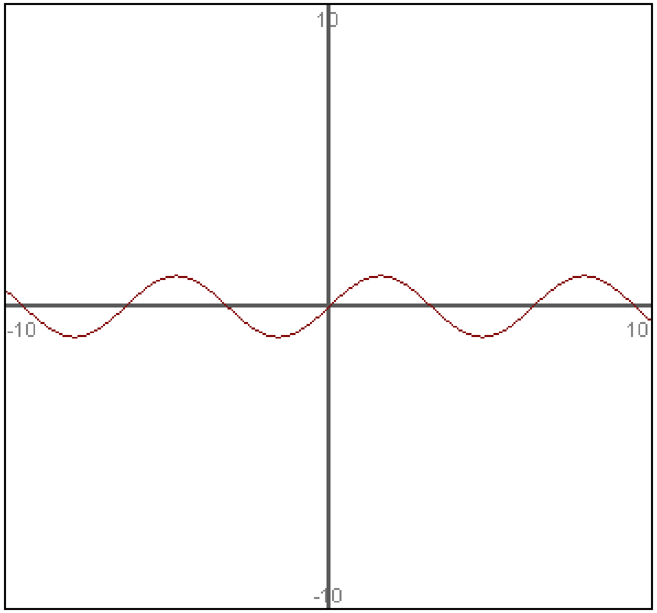
$$2. \; n(x) = \begin{cases} 2, & if \; x \leq 1 \\ x, & if \; x > 1 \end{cases}$$

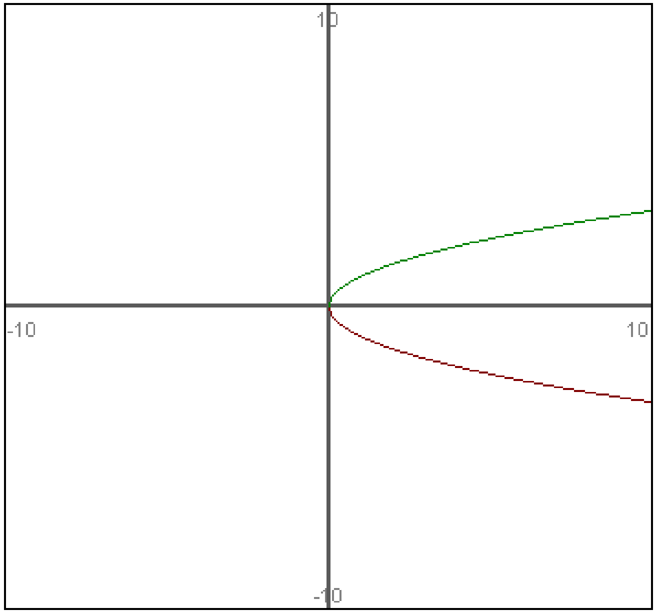
The Vertical Line Test for Functions

- DEFINITION 7.

  A graph is the **graph of a function** if and only if no vertical line intersects the graph at more than one point.

EXAMPLE 9. Which graph represents a function?





Domain of a Function

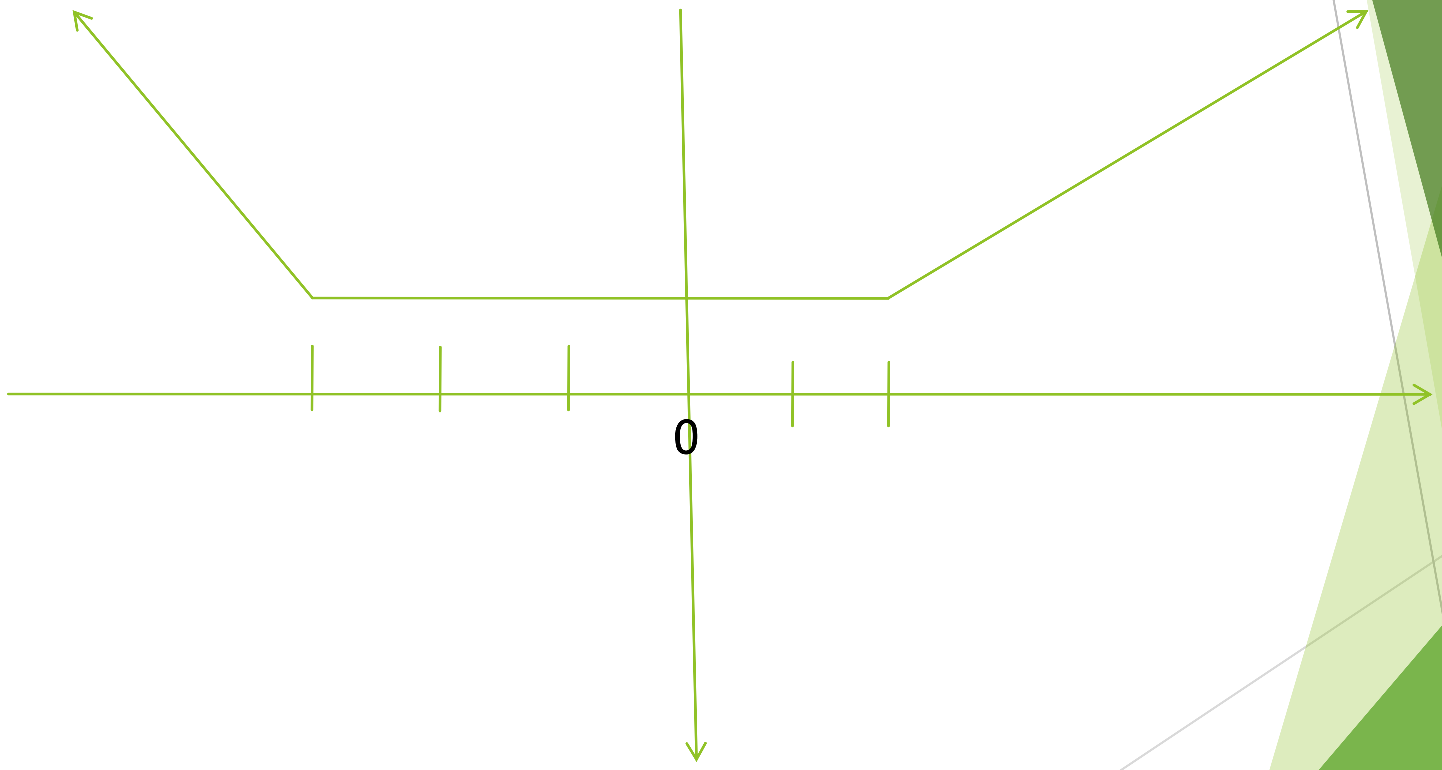
- DEFINITION 8. Unless otherwise stated, the domain of a function is the set of all real numbers for which the function makes sense and yields real numbers.
- EXAMPLE 10. Determine the domain of each function.

1. $g(t) = \frac{1}{t-4}$
2. $f(x) = \sqrt{x+1}$

Increasing, Decreasing and Constant Functions

- DEFINITION 9. If **a** and **b** are elements of an interval **I that** is a subset of the domain of a function **f,** then
- f is **increasing** on I if f(a) < f(b) whenever a < b.
- f is **decreasing** on I if f(a) > f(b) whenever a< b.
- f is **constant** on I if f(a) = f(b) for all a and b.

EXAMPLE 11. Determine the interval where the graph is increasing, decreasing or constant

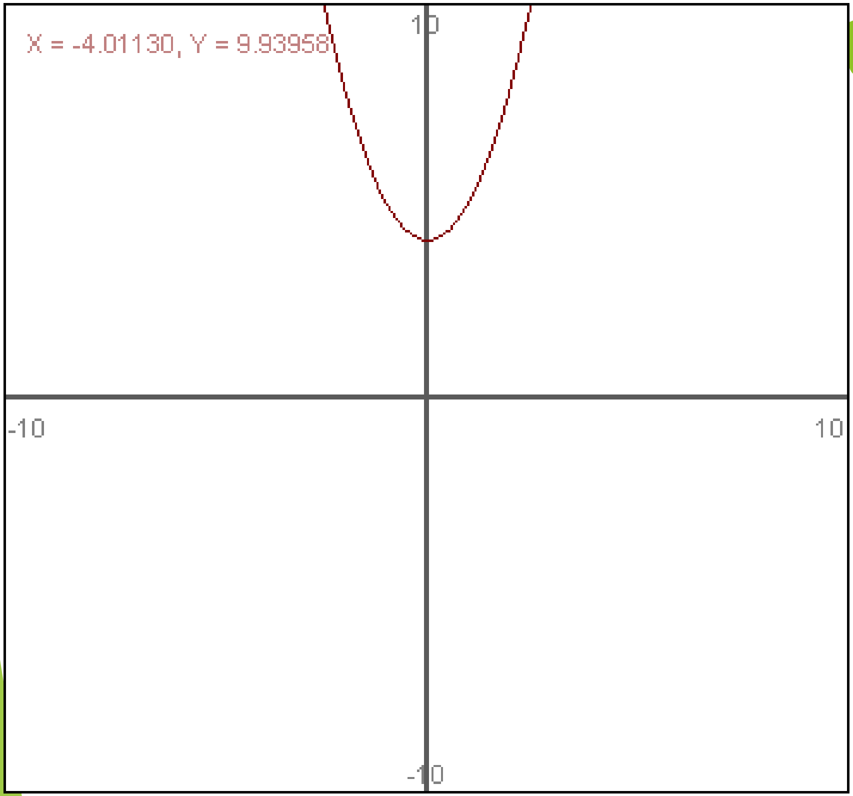


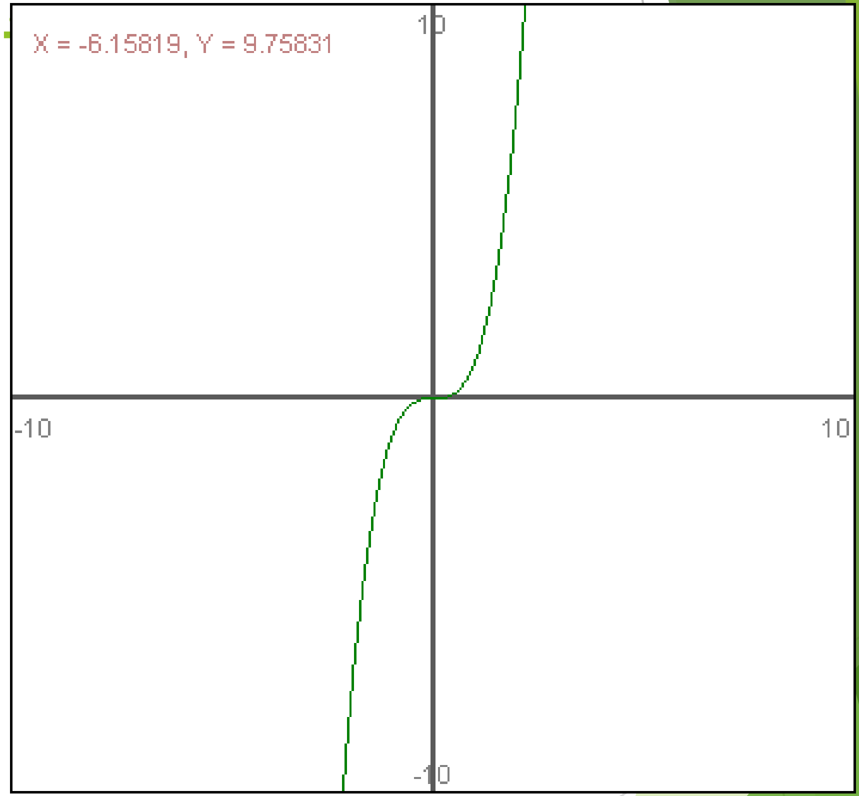
Horizontal Line Test

- DEFINITION 10.

  If every horizontal line intersects the graph of a function at most once, then the graph is the graph of a **one-to-one** function.

EXAMPLE 12. Which is the graph of a one-to-one





1.2 PROPERTIES OF GRAPHS OF FUNCTIONS

Test for Symmetry with Respect to a Coordinate Axis

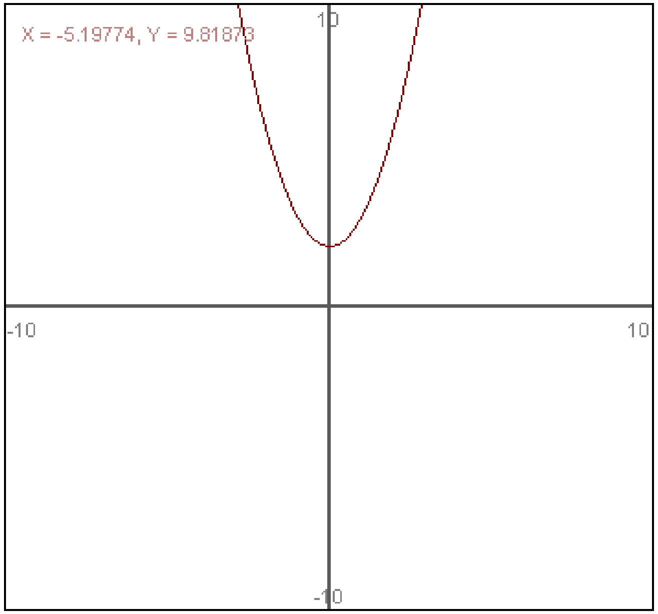
The graph of an equation is symmetric with respect to

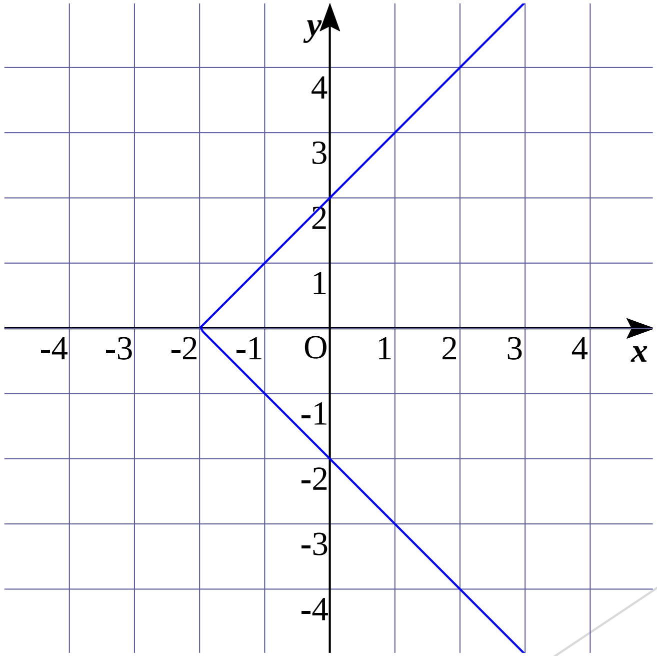
- the y-axis if the replacement x with –x leaves the equation unaltered.
- the x-axis if the replacement of y with –y leaves the equation unaltered.

Determine whether the graph of the given equation has symmetry with respect to either x-or y-axis.

1. $y = x^2 + 2$

2. $x = |y| - 2$





Symmetry with Respect to a Point

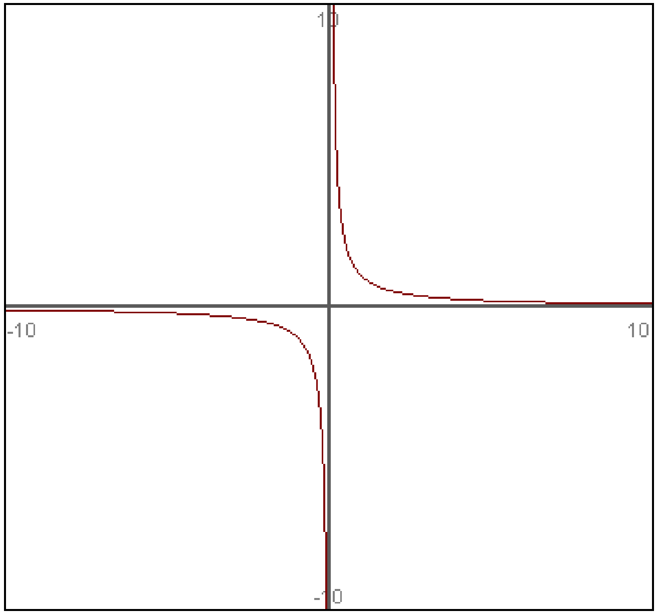
DEFINITION 1.

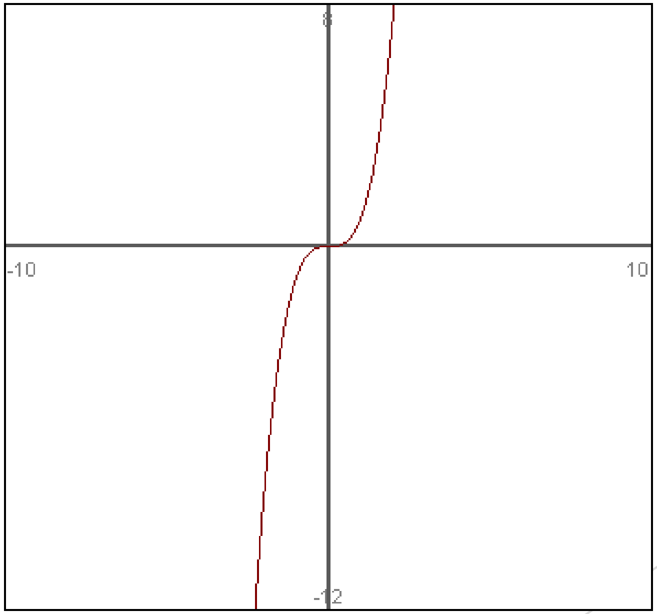
A graph is symmetric with respect to a point Q if for each point P on the graph there is a point P' on the graph such that Q is the midpoint of the line segment PP'.

Determine whether the graph of the given equations has symmetry with respect to a point. Name the point.

1. $y = \frac{1}{x}$

2. $y = x^3$





Test for Symmetry with Respect to the Origin

- The graph of an equation is symmetric with respect to the origin if the replacement of

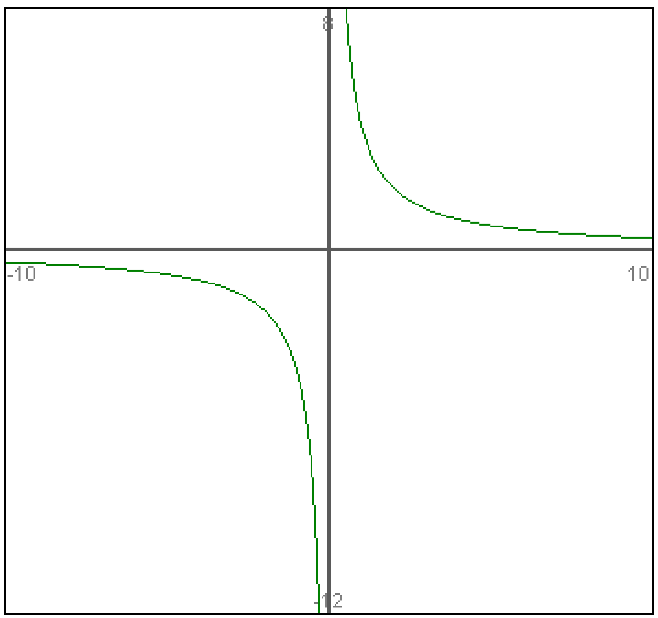
  x with –x and y with –y

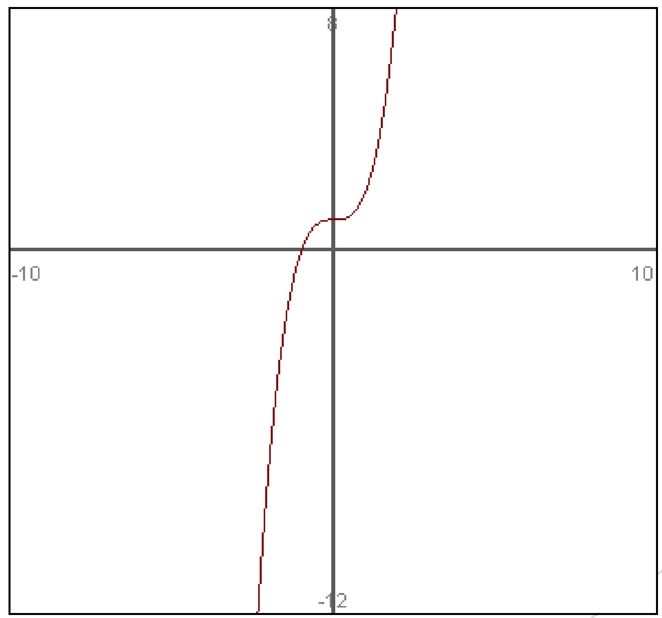
  leaves the equation unaltered.

Determine whether the graph of the given equations has symmetry with respect to the origin.

1. $xy = 4$

2. $y = x^3 + 1$





Even and Odd Function

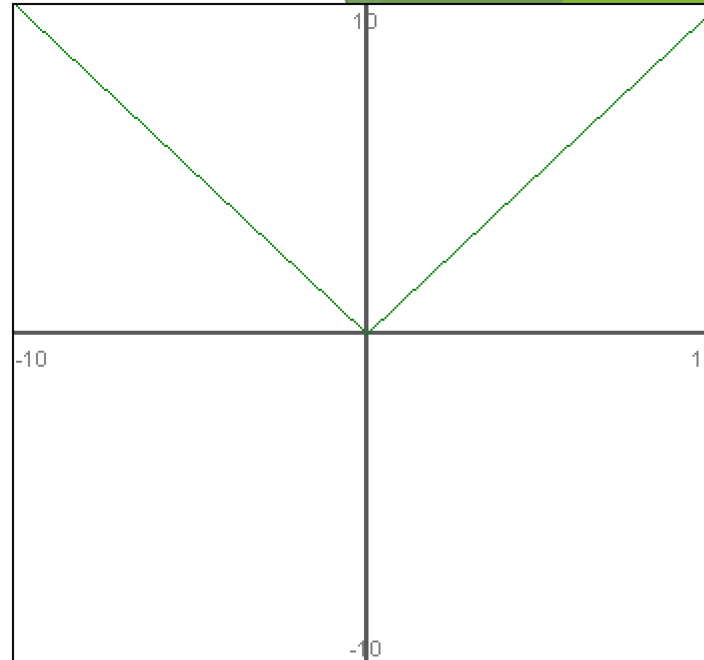
DEFINITION.

- The function f is an **even function** if and only if **f(-x) = f(x)** for all x in the domain of f.
- The function f is an **odd function** if and only if **f(-x) = -f(x)** for all x in the domain of f.

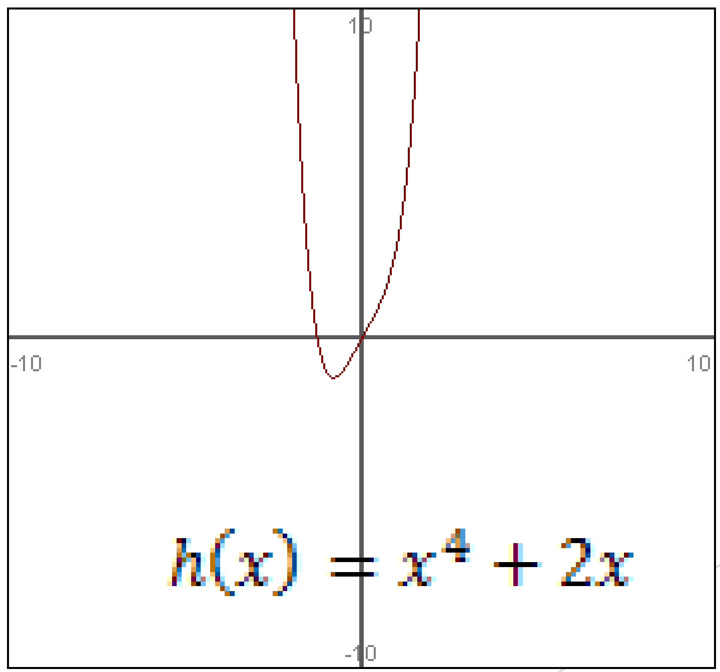
Determine whether each function is even, odd or neither.

1. $\mathrm{f}(x) = x^3$
2. $g(x) = |x|$
3. $h(x) = x^4 + 2x$

$$f(x) = x^3$$



$$f(x) = |x|$$



$$h(x) = x^4 + 2x$$

Vertical Translations

DEFINITION. If f is a function and c is a positive constant, then the graph of

- $y = f(x) + c$ is the graph of $y = f(x)$ shifted up vertically c units.
- $y = f(x) - c$ is the graph of $y = f(x)$ shifted down vertically c units.

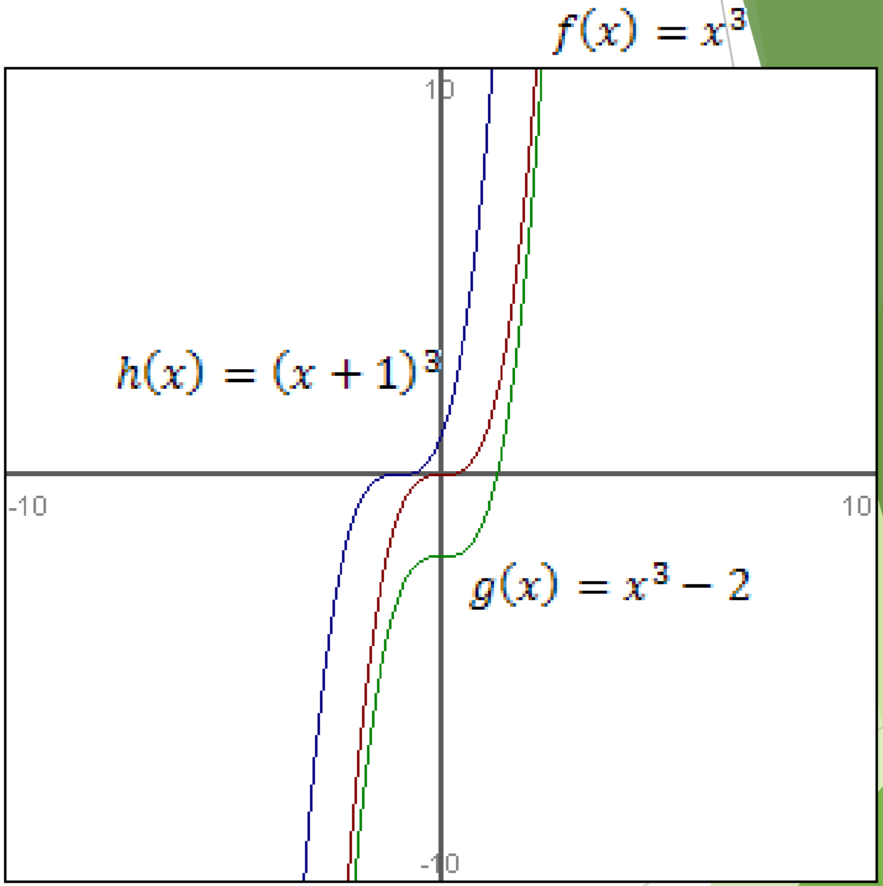
Horizontal Translations

DEFINITION. If f is a function and c is a positive constant, then the graph of

- $y = f(x - c)$ is the graph of $y = f(x)$ shifted right horizontally **c** units.
- $y = f(x + c)$ is the graph of $y = f(x)$ shifted left horizontally **c** units.

Use vertical and horizontal translation of $f(x) = x^3$ to graph the following

1. $g(x) = x^3 - 2$
2. $h(x) = (x + 1)^3$

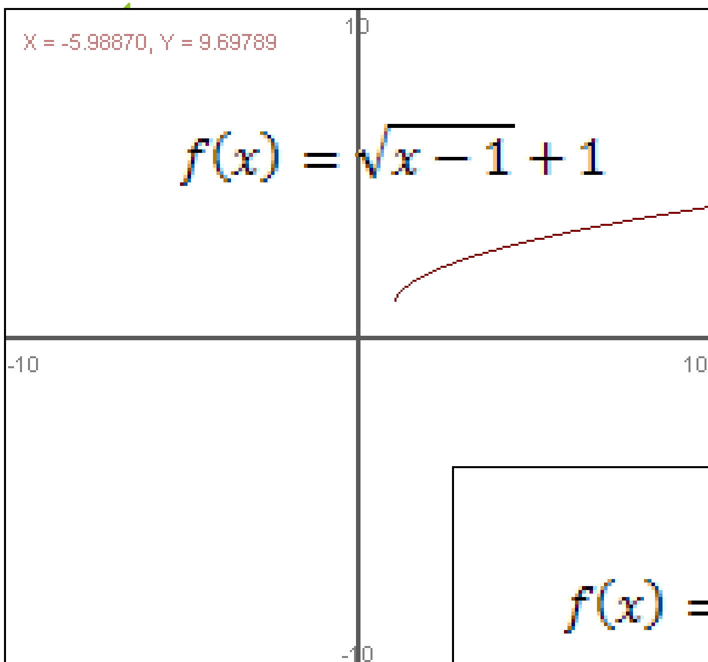


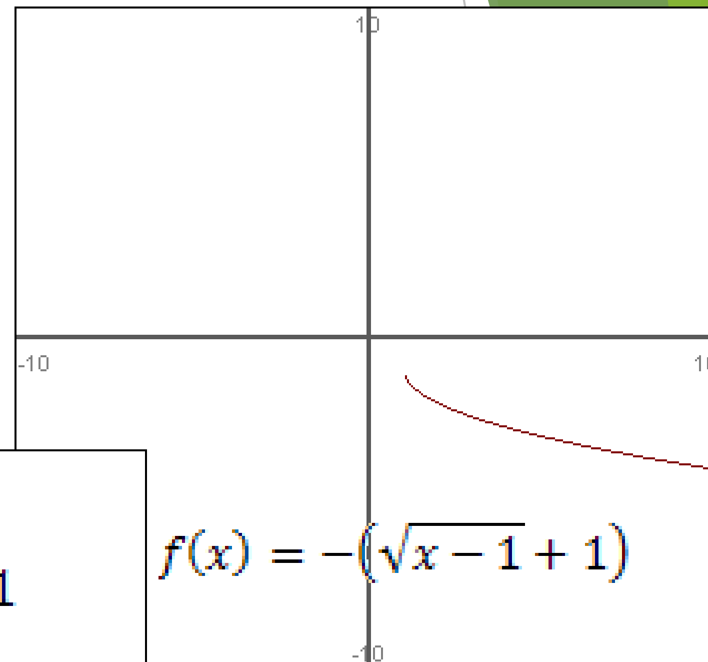
Reflections

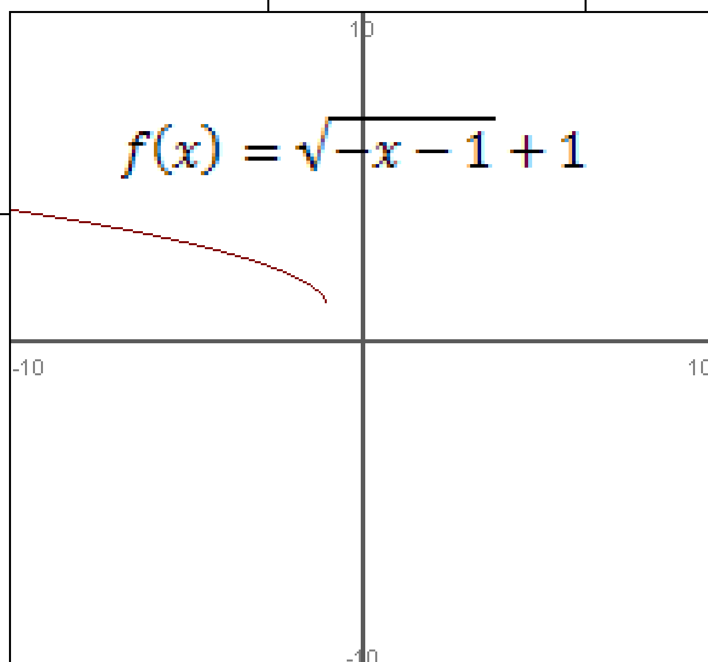
DEFINITION. The graph of

- $y = -f(x)$ is the graph of $f(x)$reflected across the x-axis.
- $y = f(-x)$ is the graph of $f(x)$reflected across the y-axis.

Use reflections of the graph of $f(x) = \sqrt{x-1} + 1$ to graph $f(x) = -\left(\sqrt{x-1} + 1\right)$ $f(x) = \sqrt{-x-1} + 1$







Shrinking & Stretching Graphs

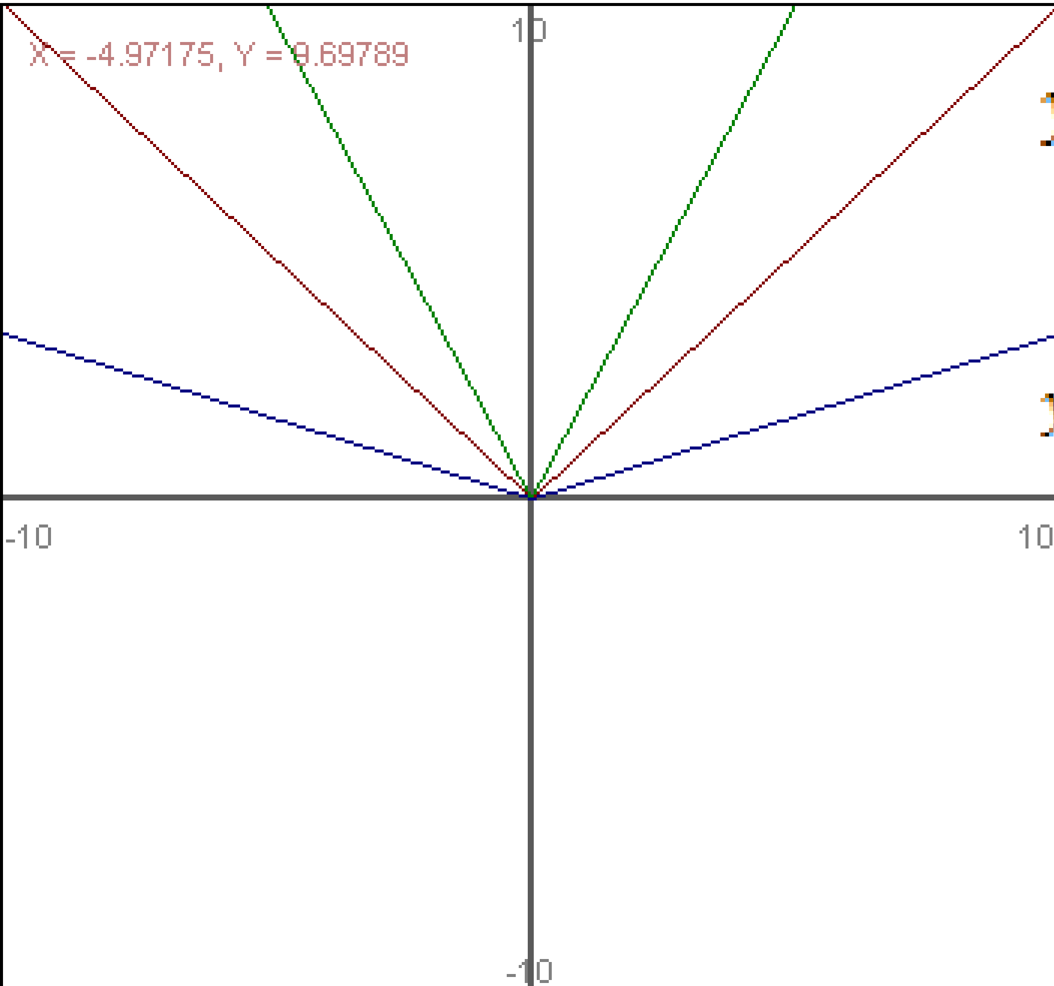
DEFINITION. Let f be a function and c a positive real number.

- If $c > 1$, the graph of $y = cf(x)$ is the graph of $y = f(x)$ vertically stretched by multiplying each of the y-coordinates by c.
- If $0 < c < 1$, the graph of $y = cf(x)$ is the graph of $y = f(x)$ vertically shrunk by multiplying each of the y-coordinates by c.

$$y = 2|x|$$

$$y = |x|$$

$$y = \frac{1}{3}|x|$$



Horizontal Shrinking & Stretching

DEFINITION. Let f be a function and **a** positive real number.

- If $a>1$, then the graph of $y = f(ax)$ is a horizontal shrinking of the graph of $y=f(x)$.
- If $0<a<1$, then the graph of $y = f(ax)$ is a horizontal stretching of the graph of $y=f(x)$.

1.3.1 TYPES OF FUNCTIONS

Types of Functions

Algebraic

Transcedental

- Trigonometric
- Inverse trigonometric
- Exponential
- Logarithmic
- Hyperbolic
- Inverse hyperbolic

Algebraic Functions

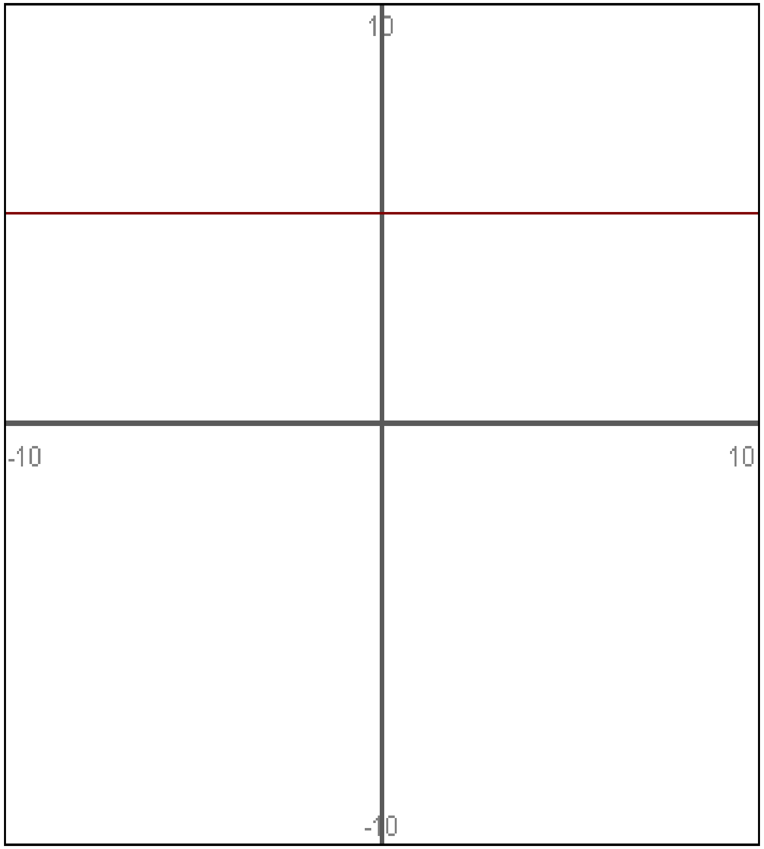
1. Polynomial
- Constant
- Linear
  - -Identity
- Quadratic
- Cubic
- n- degree

2. Rational

3.Radical
- Square root
- Cube root

5. Special Functions
- Absolute value
- Piecewise
- Greatest integer (floor)
- Ceiling
- Signum
- Unit step

Constant Function: y = c

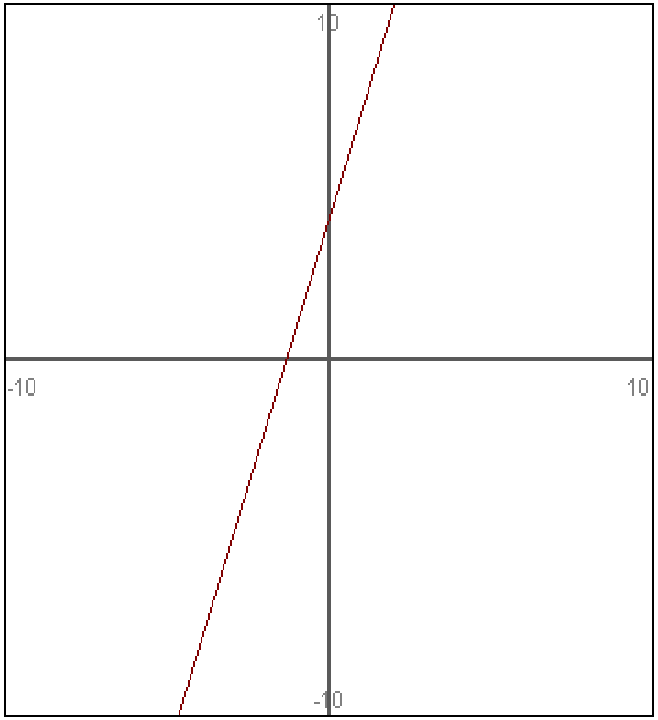


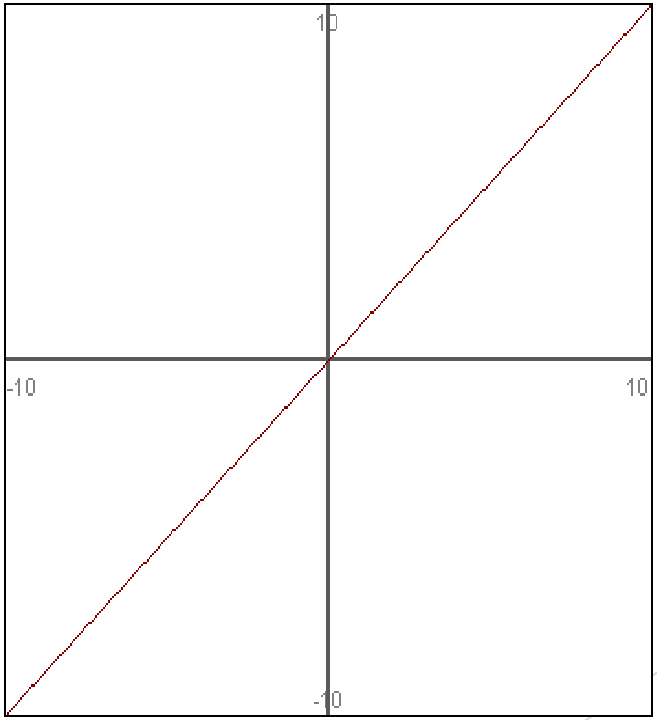
$$f(x) = 5$$

Linear Function : y = mx +b

F(x) = 3x + 4

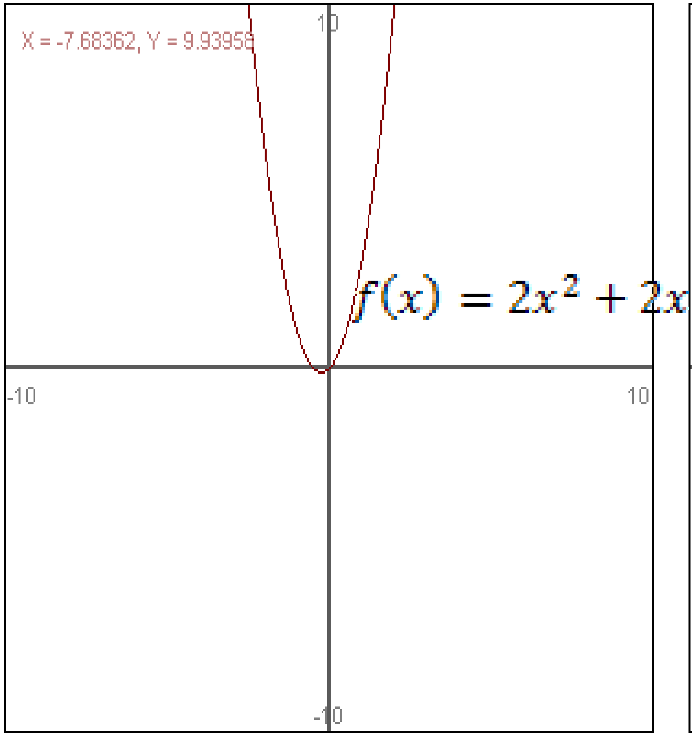
Identity Function:
G(x) = x

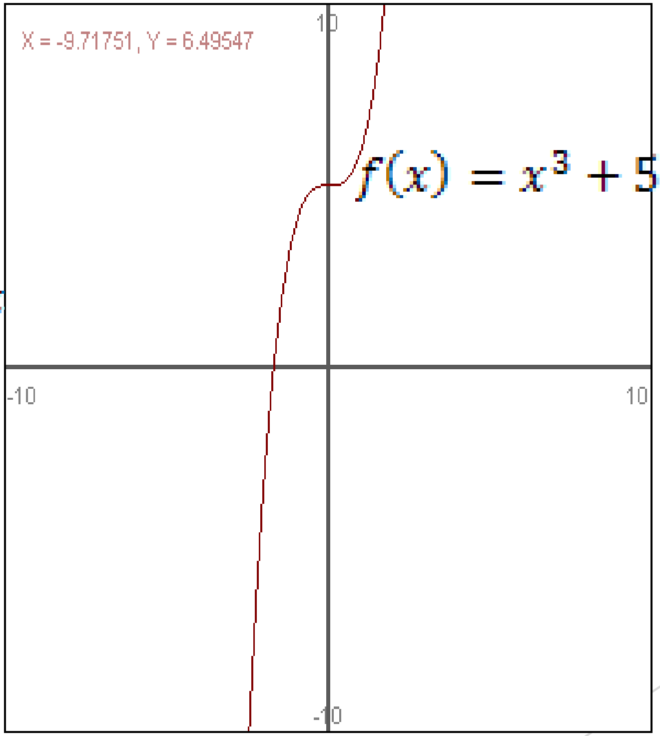




Quadratic & Cubic Functions

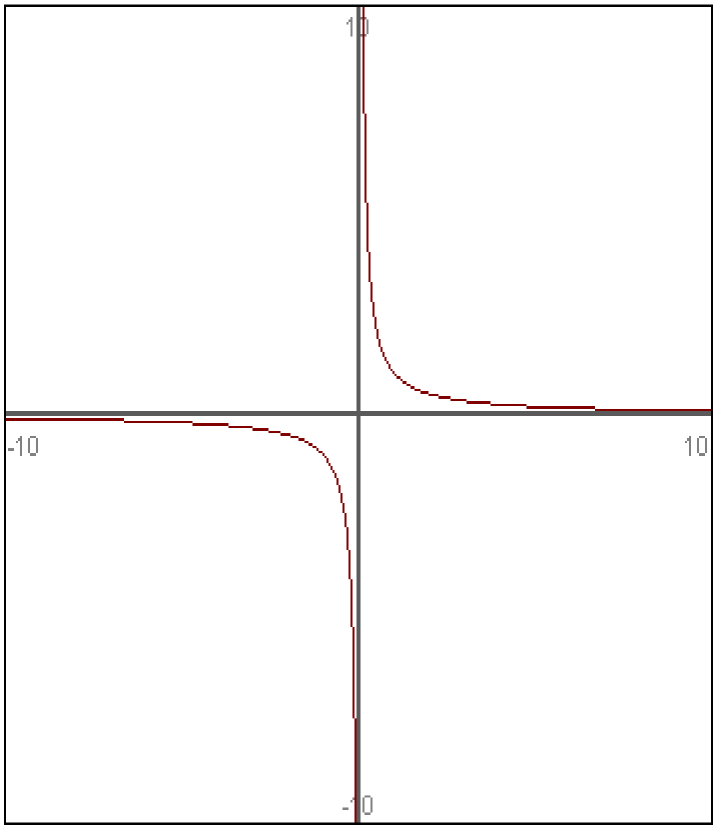
$f(x) = ax^2 + bx + c$ $\quad$ $f(x) = ax^3 + bx^2 + cx + d$



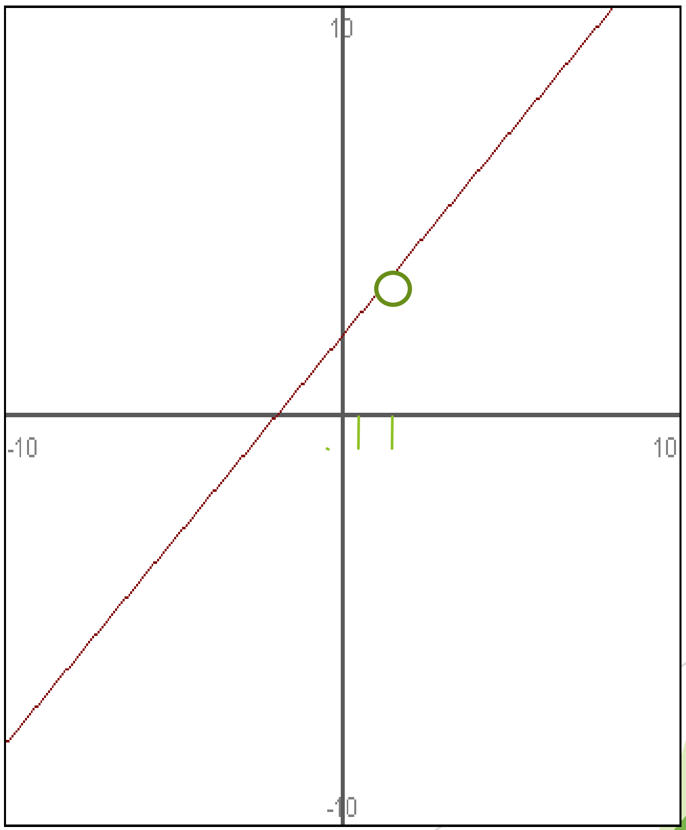


Rational Function: $f(x) = \frac{P(x)}{Q(x)}$

$$y = \frac{1}{x}$$



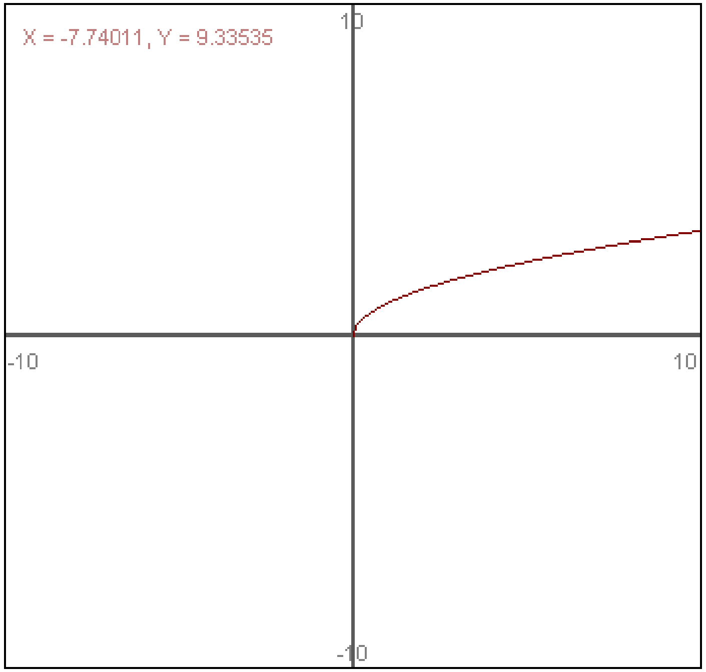
$$f(x) = \frac{x^2 - 4}{x - 2}$$

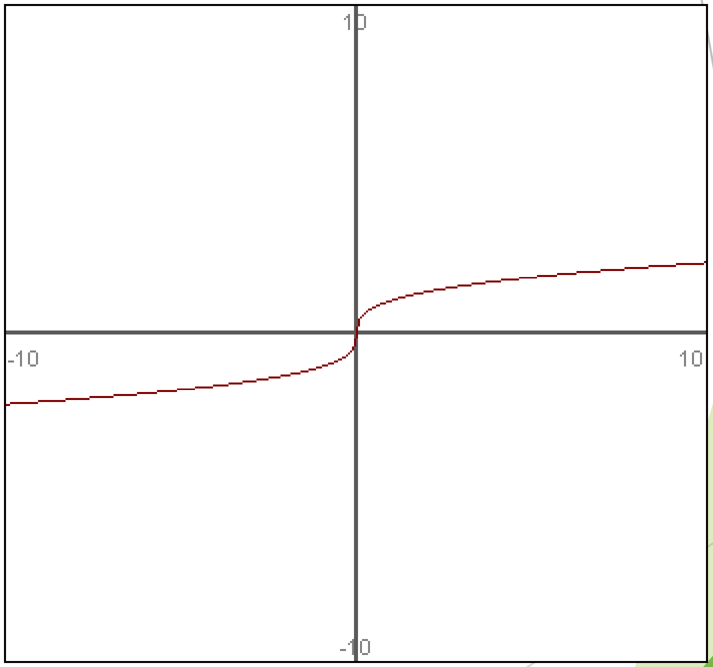


RADICAL FUNCTIONS
Square Root & Cube Root

$$f(x) = \sqrt{x}$$

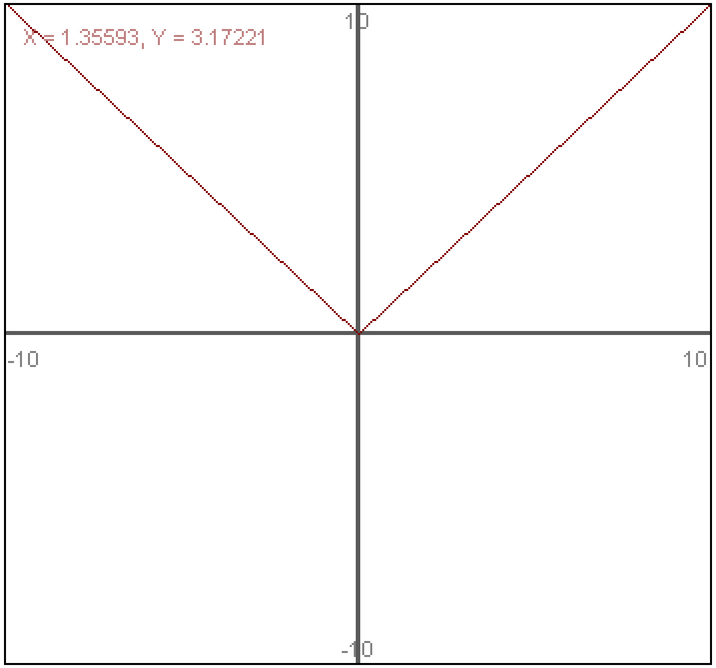
$$h(x) = \sqrt[3]{x}$$





Absolute Value Function:

$$f(x) = |x|$$



$$|x| = \begin{cases} x \,, if\ x > 0 \\ 0, if\ x = 0 \\ -x, if\ x < 0 \end{cases}$$

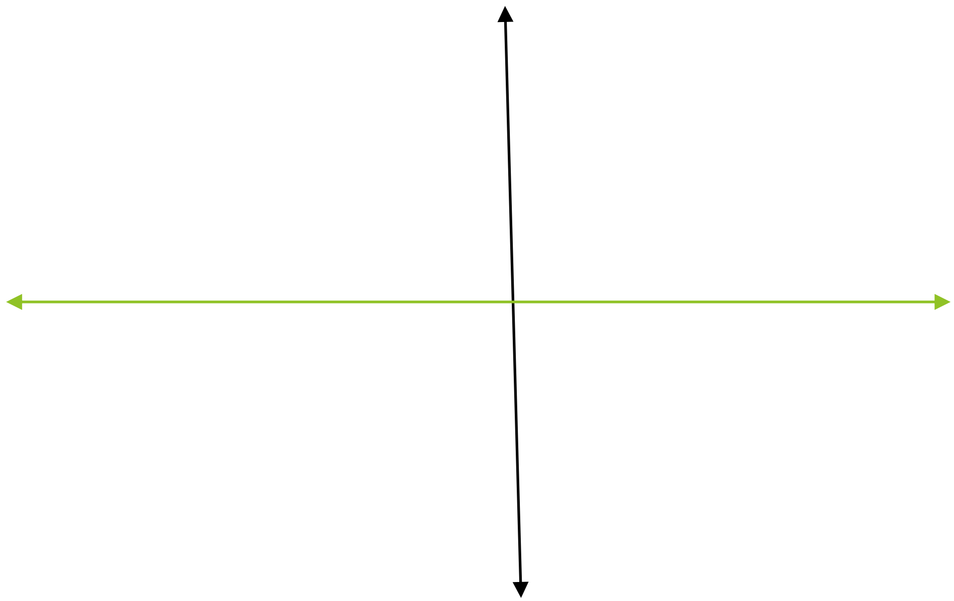
Greatest Integer Function or Floor Function: $f(x) = [\![x]\!]$

$$[\![x]\!] = n, \quad n \leq x < n + 1$$

Ceiling Function: $f(x) = \lceil x \rceil$

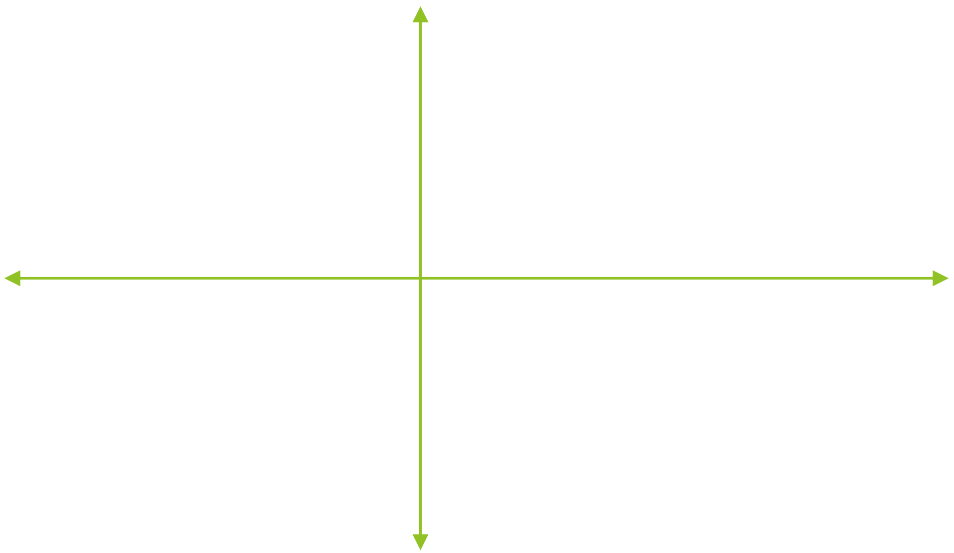
$$\lceil x \rceil = n, \qquad n < x \le n + 1$$

Signum Function: $f(x) = sgn(x)$



$$sgn(x) = \begin{cases} -1 & , x < 0 \\ 0 & , x = 0 \\ 1, & , x > 0 \end{cases}$$

Unit-Step Function: f(x)= $\mu(x)$



- $\mu(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

1.3.2 Operations on Functions

Sum, Difference, Product, & Quotient of Two Functions

DEFINITION. For all values of x for both f(x) and g(x) are defined, we define the following functions:

i. SUM $(f + g)(x) = f(x) + g(x)$

ii. DIFFERENCE $(f - g)(x) = f(x) - g(x)$

iii. PRODUCT $(f \cdot g)(x) = f(x) \cdot g(x)$

iv. QUOTIENT $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

In each case the domain of the resulting function consists of those values of x common to the domains of

f and g.

EXAMPLE 1

Given f(x) = 3x + 2 and g(x) = 4 - 5x, find

a). (f + g)(x),

b). (f - g)(x),

c). $(f \cdot g)(2)$,

d). $\left(\frac{f}{g}\right)(x)$

Give the domain of each.

EXAMPLE 2

Given f(x) = 2x, g(x) = x + 4, and $h(x) = 5 - x^3$, find

a). (f + g)(2),

b). (h - g)(2),

c). $(f \cdot h)(2)$,

d). $\left(\frac{h}{g}\right)(2)$

The Difference Quotient

DEFINITION. The expression

$$\frac{f(x+h)-f(x)}{h}, \quad h \neq 0$$

is called the **difference quotient of f.**

This function enables us to study the manner in which a function changes in value as the independent variable changes.

EXAMPLE 3

Determine the difference quotient of

$$f(x) = x^2 + 7$$

Composition of Functions

DEFINITION. For the two functions f and g, the composite function , denoted by $f \circ g$ is defined by $f \circ g$ = f(g(x))

and the domain of $f \circ g$ is the set of all numbers x in the domain of g such that g(x) is in the domain of f.

EXAMPLE 4

If $f(x) = x^2 - 3x$, and g(x) = 2x + 1, find

a). $(f \circ g)(x)$,

b). $(g \circ f)(x)$.

Inverse Function

DEFINITION.

If **f** is function with domain X and range Y, and g is a function with domain f is a one-to-one function, then g is the inverse of f if and only if

$$(f \circ g)(x) = x \; for\; all\; x\; in\; the\; domain\; of\; g, and$$

$$(g \circ f)(x) = x \; for\; all\; x\; in\; the\; domain\; of\; f$$

EXAMPLE 5

Show that $g(x) = \frac{1}{2}x - 4$ is the inverse function of f(x)= 2x+8.

Finding the inverse of the Function

To find the inverse f^{-1} of the **one-to-one** function f,

1. Substitute y for f(x)
2. Interchange x and y
3. Solve for y in terms of x
4. Substitute $f^{-1}(x)$ for y.

EXAMPLE 6

Find the inverse of the following functions:

a). $g(x) = \dfrac{x+2}{3}$

b). $h(x) = \dfrac{2x}{x+3}$

c). $f(x) = (x-2)^3$

Symmetry Property of f and f^{-1}

DEFIINITION.

The graph of a function **f** and and the graph of its inverse $\mathbf{f}^{-1}$ are symmetric with respect to the line

y = x.

EXAMPLE 7

Sketch the graph of inverse of f.

(use the symmetry property)

a). $f(x) = x^2, \ x \geq 0$

b). $f(x) = x^2, \ x \leq 0$

c). $f(x) = x^2 - 4x, \ x \geq 2$

1.4 Functions as Mathematical Models

Suggestion for Solving Problems Involving a Function as a Mathematical Model

1. Read the problem carefully so that you understand it.
2. Determine the known and unknown quantities.
3. Write down numerical facts. You may use variation or proportion or a formula.
4. From the information in (3), determine the algebraic expression which is a function as a mathematical model of the problem.
5. Apply the mathematical model to solve for the unknown quantities and write a conclusion that answers the question of the problem.

EXAMPLE 1: 2/27

A person's approximate brain weight is directly proportional to his or her body weight, and a person weighing 150 lb has an approximate brain weight of 4 lb.

(a) Find a mathematical model expressing the brain weight of a person as a function of the person's body weight. $w_{BR} = \frac{2}{75} w_{BO}$

(b) Find the approximate brain weight of a person whose body weight is 176 lb.

EXAMPLE 2: 6/27

In 1995, the postage of a domestic first-class letter was computed as follows: 32 cents for the first ounce or less, then 23 cents for each ounce(or fractional part of an ounce) for the next 10 oz.

(a) Find a mathematical model expressing the postage of a first class letter, weighing not more than 11 oz, as a function of its weight.

(b) Sketch the graph of your function in (a).

(c) Determine the postage of weights 1.6 oz, 2 oz, 8.4 oz, and 11 oz.

EXAMPLE 3: 10/27

In a lake a large fish feeds on a medium-size fish and the population of the large fish is a function f of x, the number of medium –size fish in the lake. In turn the medium-size fish feed on small fish, and the population of the medium-size fish is a function of w, the number of small fish in the lake. If

$$f(x) = \sqrt{20x} + 150 \quad and\ g(w) = \sqrt{w} + 5000$$

(a) Find a mathematical model expressing the population of the large fish as a function of the number of small fish in the lake.

(b) Find the number of large fish when the lake contains 9 million small fish.

EXERCISES

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